IIT GOA

B.Tech.
Mathematics and Computing
2019

Overview

This course will provide students with a rigorous background in mathematics and computing. The aim is to lay a strong foundation for applying mathematical and computational insights in solving problems in domains ranging from scientific modelling to machine learning. Graduates of this programme will be equipped to take up research and development positions in fields such as any branch of Mathematics, Scientific Computing, Optimization, Machine learning, Data Analytics etc.

In addition, by an appropriate choice of electives, students will have adequate background and basic training required for graduate studies in the subject of their choice, namely Mathematics or Computer Science. Any such specialization will be indicated as "Honours/ Minors" in the degree.

Program Description

The IIT GOA Curriculum Committee will provide a template for each of the BTech Programs. The first two semester courses are considered the Core, and all courses will be common for students in any branch.

In the first semester, IIT GOA is starting a zero-credit P/F course called Introduction to Profession. Through this course, it is proposed to tell students what lays ahead, the excitement and the prospects and more.

In the following pages we give the syllabi of all the mandatory courses pertaining to the degree BTech in Mathematics and Computing, starting after the Core. The assignement of semesters may be considered tentative, and credit assigments of these courses will be updated shortly.

The Elective courses are not listed here and will be developed (specially designed for the students' interests). Across the semesters there will be at least 7 elective courses (Department Electives and Open Electives). Two electives can be converted to a B Tech Project allowing for specialization in an area of choice in either Mathematics or Computer Science.

Course Syllabi:

III.1 Real Analysis.
III.2 Discrete Mathematics.
III.3 Probability and Statistics.
III.4 Linear Algebra.

IV.1 Design and Analysis of Algorithms.
IV.2 Multivariate Calculus.
IV.3 Numerical Analysis. (to V)
IV.4 Algebra.
+ ELECTIVE

V.1 Stochastic Processes.
V.2 Complex Analysis.
V.3 Theory of Computation.
+ 2 ELECTIVES (OE and DE)

VI.1 Differential Equations.
VI.2 Topology and Geometry.
VI.3 Machine Learning.
+ 2 ELECTIVES (OE and DE)

VII.1 Functional Analysis.
+ ELECTIVES and BTech Project

VIII : ELECTIVES and BTech Project

III.1 REAL ANALYSIS

Review: The real number system; Archimedian property, Completeness, Convergence of sequences and series, limits. Continuity, Uniform Continuity.

Metric Spaces. Introduce distance, define metric. Metric spaces, Examples $\mathbb{R}^n$, ℓ^2, l^p (Holder and Minkowski inequalities), $C[a, b]$ (Uniform Convergence). Open sets, closed sets, and examples of these in different metrics.cantor set. Complete Metric Spaces, completeness of $C[a, b]$. Compactness, with many examples. Finite Intersection property. Compact subsets in $\mathbb{R}^n$. Brouwers Fixed point Theorem in $\mathbb{R}^2$. (Applications). Connectedness, IVP, path connectedness. Continuous functions on connected sets. (Applications)

Differentiation. Derivatives of functions, Taylors theorem. Monotonic functions, Functions of bounded variation; Absolutely continuous functions.

Reimann Integration. Properties of Riemann integral, characterization of Riemann integrable functions. Improper integrals. Pointwise convergence, uniform convergence of functions, relation with convergence of functions in the mean, differentiation, integration; Examples.

Polynomial Approximations Power series, Taylor series. Weierstrass Approximation Theorem, Bernstein Polynomials. Fourier Series, computation of Fourier coefficients; smoothness and decay. Different kinds of convergence. Fejers Theorem (averaging). Another proof of Weierstrass Theorem. Rates of convergence and comparison. (Use MATLAB...)

Suggested References.

- T. Apostol, Mathematical Analysis.
- W. Rudin, Principles of Mathematical Analysis.
- Terence Tao, Analysis I and II. Trim Series.
- R. R. Goldberg: Methods of Real Analysis.
- N.L. Carothers, Real Analysis.
- Kenneth R. Davidson and Allan P. Donsig, Real Analysis and Applications- Theory and Practice.

III.2 DISCRETE MATHEMATICS

Logic. What is a proof? proof strategies — proof by contradiction, contrapositive, induction, direct proofs, examples from elementary number theory, graph theory. Some examples of fallacious proofs.

Propositional logic: propositional connectives, syntax of propositional formulas, semantics, truth tables, tautologies, satisfiability, contradictions, logical implications, functional completeness of propositional connectives, Proof that negation cannot be expressed using AND and OR.Formal proofs using natural deduction/ semantic tableau. Notion of soundness and completeness (if we do semantic tableau system, then we can also do its soundness and completeness proofs). normal forms: CNF and DNF, converting formulas in to normal forms.

Language of first-order logic, expressing natural language statements in the language of first order logic. examples from arithmetic and graphs, syntax, quantification, informal understanding of semantics, sample validities, prenex normal form, converting formulas to prenex normal form.

Sets and Relations. Sets, examples and non-examples, defining sets, set building operations, various identities and how to prove them, Venn diagram, size of sets, finite and infinite sets, countable sets, countability of rationals, algebraic numbers, polynomials, finite sequences etc., Cantors diagonalisation and uncountability of reals, irrationals, discontinuum.

Schroeder-Bernstein Theorem (proof could be optional) Relations, equivalence relations, correspondence between equivalence relations and set partitions, quotient with respect to an equivalence relation, order relations, partial orders, Dilworths theorem (proof could be optional), lattices, boolean algebras (optional)

Combinatorics Law of sum and law of products in counting, Permutations and combinations, arrangement/selection of distinct/identical objects, enumerating set and integer partitions, Stirling numbers, counting techniques, pigeonhole principle, recurrence relations, generating functions and their use in solving recurrence relations, Burnsides Lemma, Polyas counting theorem (after a very short introduction to finite groups), Principle of inclusion and exclusion.

Graph Theory Graphs, representations using matrices, trees definitions, basic properties, connectivity, paths, cycles, Eulerian walks, Hamiltonian cycles, Eulers theorem, cliques, colourings, chromatic index, various bounds, chromatic polynomial, graph matching, Halls theorem, planarity, Eulers formula, Kuratowskis theorem,

Suggested References:
C.L. Liu. Elements of Discrete Mathematics.
Rosen, Discrete Mathematics for Computer Science

III.3 PROBABILITY AND STATISTICS.

Probability: (uncertain world, perfect knowledge of the uncertainty).Discrete experiments: Motivation; equally likely outcomes and combinatorial problems (matching, birthday problem, balls, and boxes); non-equally likely outcomes, conditional probability, Urn models, Bayes formula, Independent events, repeated trials. Random variables and distributions: Bernoulli, Binomial, Poisson (also as the limit of Binomials), geometric (also as waiting time), negative binomial and hypergeometric. expectation, variance, moments. Joint distributions, sums of independent random variables (by moment generating functions and by convolution), correlation. Conditional distributions and conditional expectation. Chebyshev, Weak Law of Large Numbers (WLLN), statement of Central Limit Theorem (CLT).

Continuous variables: Density functions, uniform, exponential, beta, gamma, normal and Cauchy variables; expectation, variance and moments, Independent variables, sums of independent random variables, Joint densities, correlation, Multivariate normal (matrix manipulations, marginals, linear transformations), Dirichlet distribution. Conditional densities and conditional expectation. Chebyshev, WLLN patterns of decimal digits, statement of CLT.

Statistics: applied probability (data in an uncertain world, perfect/imperfect knowledge of the uncertainty) Frequentist significance tests and confidence intervals. Resampling methods, bootstrapping. Linear regression. Computation, simulation, and visualization using R will be used throughout the course.

Suggested References.

- P G Hoel, S C Port, C J Stone: Introduction to probability theory
- P G Hoel, S C Port, C J Stone: Introduction to statistical theory.
- P. Bremaud: Discrete Probability Models and Methods.
- H M Taylor, S Karlin: An introduction to stochastic modeling.

- **D P Bertsekas, J N Tsitsiklis: Introduction to Probability (MIT notes).**
- **W Feller: An Introduction to probability theory and its applications. (Vols. 1, 2).**

Remarks.

Students completing the course will be able to:
- Use R to run basic simulations of probabilistic scenarios.
- Create and interpret scatter plots and histograms.
- Understand the difference between probability and likelihood functions, and find the maximum likelihood estimate for a model parameter.
- Find credible intervals for parameter estimates.
- Use null hypothesis significance testing (NHST) to test the significance of results, and understand and compute the p-value for these tests.
- Use specific significance tests including, z-test t-test (one and two sample), chi-squared test.
- Find confidence intervals for parameter estimates.
- Use bootstrapping to estimate confidence intervals.
- Compute and interpret simple linear regression between two variables.
- Set up a least squares fit of data to a model.

Note: Approximately one-third time on each of the three parts.

III.4 LINEAR ALGEBRA. 8 credits

Notion of a field. Vector spaces over fields, subspaces, bases and dimension.

Linear transformations. Representation of linear transformations by matrices, effect of change-of-basis, rank-nullity theorem, Gaussian elimination revisited with application to determination of rank, bases for row-space, column-space of a matrix, and solution space of a corresponding system of homogeneous linear equations. Applications to graphs and networks. Elementary matrices, LU decomposition. Equivalence of matrices.

Eigenvalues and eigenvectors, characteristic polynomials, minimal polynomials, Cayley- Hamilton Theorem, triangulation, diagonalization, matrix exponentials, rational canonical form, Jordan canonical form.

Positive definite matrices, minorant characterization, Singular value decomposition. Finite element method. Bilinear forms, symmetric and skew-symmetric bilinear forms, real quadratic forms, Sylvester's law of inertia.

Applications: Page-rank algorithm. Linear programming, Network models, game theory.

Suggested References:

- **G. Strang, Linear Algebra and its Applications.**
- **D. C. Lay, Linear Algebra and its Applications.**
- **K. Hoffman and R. Kunze, Linear Algebra.**
- **S. Lang, Linear Algebra.**
- **P. Lax, Linear Algebra and its Applications.**

IV.1 DESIGN AND ANALYSIS OF ALGORITHMS.

Models of computation, algorithm analysis, time and space complexity, average and worst case analysis, lower bounds.

Algorithm design techniques: divide and conquer, greedy, dynamic programming, amortization, randomization. Examples of algorithms illustrating these techniques including Quicksort, Minimum Spanning Trees, Integer and Matrix Multiplication, Fast Fourier Transform. Reductions. Problem classes P, NP, NP-hard and NP-complete problems. Approximation algorithms for some NP-hard problems.

Suggested References.

- **Algorithm Design, by J. Kleinberg and E. Tardos, Addison-Wesley, 2005.**
- **Introduction to Algorithms (3rd Edition), by T.Cormen, C. Leiserson, R. Rivest, and C. Stein, The MIT Press, 2009.**
- **Algorithms, by S. Dasgupta, C. Papadimitriou, and U. Vazirani, McGraw- Hill, 2006.**

IV.2 MULTIVARIATE CALCULUS

Cartesian and Polar coordinate systems for $\mathbb{R}^n$, Volume element.

Functions of several variables, Continuity. Several examples.

Differentiation. Partial derivatives and the Tangent space. The Chain Rule. Inverse function Theorem and Implicit function Theorem.

Higher Derivatives. Extrema of Functions in several variables.

Applications. Planetary Motion.

Review of Reimann Integration on $\mathbb{R}^n$. Iterated Integrals, change of variables, Jacobian.

Line integrals, Surface Integrals. Green's Theorem, Divergence Theorem, Stokes' Theorem.

Applications; Conservation Laws.

References.

- **P.D. Lax and M.S. Terrell. Multivariate Calculus with applications. 2017. (Course Textbook)**
- **M. Spivak. Calculus on Manifolds.**
- **J.R. Munkres. Manifolds.**

IV.3 NUMERICAL ANALYSIS

System of linear equations: matrix norms, Gaussian Elimination, LU decomposition, QR decomposition, Gauss Jacobi and Gauss Seidel methods with convergence analysis, condition number, Gershgorin theorem for locating eigenvalues, power method to approximate the eigenvalues.

Nonlinear equations/systems: Bisection method, Regula Falsi, Secant method, Newtons method, fixed point iteration and order of convergence.

Interpolation/Approximation: Polynomial interpolation, Hermite interpolation, spline interpolation, error analysis. Numerical Integration: Trapezoidal and Simpsons rules, Gaussian quadrature formulae and error analysis.

Numerical Differentiation: Forward, backward and central difference approximations, single and multistep methods for initial value problems.

Fourier Analysis and Applications: Fourier series , Fourier transform,, The Discrete Fourier Transform and its algorithm: the Fast Fourier Transform, Spectral Interpolation, Differentiation, Quadrature.

Suggested References:

- **S. D. Conte and Carl de Boor, Elementary Numerical Analysis- An Algorithmic Approach.**
- **K. E. Atkinson, Introduction to Numerical Analysis.**
- **Steven C. Chapra, Applied Numerical Methods with MATLAB for Engineers and Scientists.**
- **D. Kincaid and W. Cheney, Numerical Analysis: Mathematics of Scientific Computing. (2002).**
- **G. B. Folland, Fourier analysis and its applications.(2009).**
- **Lloyd N. Trefethen, Spectral Methods in MATLAB, SIAM (2000).**

IV.4 ALGEBRA

Rudiments of rings and fields, elementary properties, polynomials in one and several variables, divisibility, irreducible polynomials, Division algorithm, Remainder Theorem, Factor Theorem, Rational Zeros Theorem, Relation between the roots and coefficients, Newton's Theorem on symmetric functions, Newton's identities, Fundamental Theorem of Algebra (Statement and equivalent versions). Rational functions, partial fraction decomposition. Resultants and discriminants.

Groups, subgroups and factor groups, Lagrange's Theorem, homomorphisms, normal subgroups. Quotients of groups. Basic examples of groups: symmetric groups, matrix groups, group of rigid motions of the plane and finite groups of motions.

Cyclic groups, generators and relations, Cayley's Theorem, group actions, Sylow Theorems. Direct products, Structure Theorem for finite abelian groups. Simple groups and solvable groups, nilpotent groups, simplicity of alternating groups, composition series, Jordan- Holder Theorem. Semidirect products. Free groups, free abelian groups.

Rings, Examples (including polynomial rings, formal power series rings, matrix rings and group rings), ideals, prime and maximal ideals, rings of fractions, homomorphisms and isomorphisms.

Fields, prime fields, characteristic of a field. Field extensions, finite extensions, algebraic extensions, simple extensions. Splitting field of a polynomial over a field. Applications: Classical ruler and compass constructions. Construction of finite fields. Regular polygons. Introduction to public key cryptography.

Suggested References:

- M. Artin, Algebra.
- L. Childs, A Concrete Introduction to Higher Algebra.
- J. A. Gallian, Contemporary Abstract Algebra.
- S. Lang, Undergraduate Algebra.
- S. Lang, Algebra.
- J. T. Lee, Abstract Algebra: An Introductory Course.
- J. Stillwell, Elements of Algebra. Springer, 1994.

V.1 STOCHASTIC PROCESSES

Simple symmetric random walk, recurrence in dimensions one/two and transience in three.

Finite state Markov chains, examples. recurrence, transience and aperiodicity, fundamental theorem for irreducible chains; (Renewal equation),gamblers ruin, Ehrenfest urn model, simulations, card shuffling.

Infinte state chains: Random walks, Markov chains with countable state space. criterion for recurrence, Birth and Death chains.

Poisson processes and various ways of looking (axiomatic definition; construction using exponential variables; via differential equations), Com- pound Poisson Process, M/M/1 Queue.

Brownian motion; brief introduction.

Suggested References:

- H M Taylor, S Karlin: An introduction to stochastic modelling.
- P Bremaud: Discrete Probability Models and Methods.
- Olle haggstrom; Finite Markov Chains and algorithmic applications.
- P Bremaud: Markov chains: Gibbs fields, MonteCarlo simulation and Queues.
- P G Hoel, S C Port, C J Stone: Introduction to Stochastic processes.
- D P Bertsekas, J N Tsitsiklis: Introduction to Probability (MIT notes).
- W Feller: An Introduction to probability theory and its applications Vol 1

V.2 COMPLEX ANALYSIS

Complex numbers: algebraic properties, graphical representation, Riemann Sphere, limits and continuity, Differentiability, CR equations,

Analytic functions. Elementary functions, integration on contours, Cauchys theorem, Cauchys integral formula and its applications, Moreras theorem. Series of complex numbers, Taylors theorem, Sequence of analytic functions, Schwartz reflection principle, Runges approximation theorem, Identity theorem, Maximum modulus principle, Laurent series, Cauchys Residue theorem.

Classification of isolated singularities, Riemanns theorem on removable singularities, Casorati-Wierstrass theorem.

Meromorphic functions, Argument principle, winding number, Rouches theorem and its applications, Open mapping theorem.

Evaluation of certain improper integrals, Conformal mapping, Mobius transformation, Schwartz Lemma, Picks lemma, Wierestrass theorem for infinite products.

Harmonic functions, Poisson integral formula, Mean value Property, Dirichlet problem.

Suggested References

- Churchill, Brown, Complex variables and Applications. (2009)
- Stien and Shakarchi, Complex analysis. (2013).
- Ahlfors, Complex Analysis.
- Serge Lang, Complex Analysis.
- Conway, Functions of One Complex Variable.

Additional References:

1. Youtube video on introduction to complex numbers by welchlab : https://www.youtube.com/watch?v=T647CGsuOVU.

2. Demostration of Riemann sheets : http://demonstrations.wolfram.com/ 3. Graphics of some conformal maps: http://www-users.math.umn.edu/ arnold/complex0.html

V.3 THEORY OF COMPUTATION

Finite Automata: Notion of a formal language. Deterministic finite automata (DFA): language accepted by a DFA. Nondeterministic finite automata (NFA), NFA with epsilon transitions, language accepted by an NFA, equivalence of DFAs and NFAs, class of regular languages. Regular expressions, regular expressions capture precisely the class of regular languages. Closure properties of regular languages. Pumping lemma for regular languages and its application in proving non-regularity. Myhill-Nerode theorem. Minimization of DFAs. Decision properties of regular languages.

Context Free Languages (CFL): Notion of a grammar. Context free grammars (CFG), derivations from a CFG, parse trees. Push-down automata (PDA), acceptance by final state and acceptance by final states. Equivalence of PDAs and CFGs. Chomsky normal forms of CFGs. Pumping lemma for CFLs and its applications in proving languages non-context free. Closure properties of CFLs. Decision properties of CFLs. . Push-down automata (PDA): Languages acceptance by PDAs: by empty stack and by final states and their equivalence, languages accepted by PDAs are precisely CFLs.

Turing Machine (TM): Definition. Language accepted by a TM. Robustness of TMs: extensions of basic TM model including nondeterministic TM, restrictions of basic TM model. Extensions and restrictions both capture the same class of languages, viz. recursively enumerable (r.e.) languages. Church-Turing thesis. Universal TM. Undecidability. Halting problem and its undecidability. Notion of reductions. Use of reductions in proving undecidability. Recursive languages, separation of recursive and r.e. classes.

Intractibility: Definition of classes P and NP. Extended Church Turing thesis. Polynomial time reductions. NP-hardness and NP-completeness. Cook-Levin theorem (proof if time permits). Example NP-completeness proofs by reducing to satisfiability.

Reference.

Ullman. 3rd Edition. Pearson, 2008.

VI.1 DIFFERENTIAL EQUATIONS.

ODEs: System of ordinary differential equations, Local existence and uniqueness of solutions, Global existence, Stability theory, Power series solution, Sturm-Liouville problems.

PDEs: First order PDEs- Linear and quasi linear PDE's, Methods of characteristics, Cauchy problems. Second order PDEs and their classification. Laplace, Heat and Wave equations- Fourier series, Method of separation of variables, Harmonic functions, Maximum principles, Fundamental solution.

References:

- Vladimir I. Arnold, Ordinary Differential Equations.
- Earl A. Coddington, Norman Levinson, Theory of Ordinary Differential Equations.
- Lawrence Perko, Differential Equations and Dynamical Systems. 3rd edition, 2000.
- William E. Boyce, Richard C. DiPrima, 2009. Elementary Differential Equations and Boundary Value Problems By
- Walter A. Strauss, Partial Differential Equations: An Introduction.
- Mark A. Pinsky, Partial Differential Equations and Boundary-value Problems with Applications.
- Sandro Salsa. Partial Differential Equations in Action: From Modelling to Theory, 2015.
- Lawrence C. Evans Partial Differential Equations By , 2nd edition, 2010.

VI.3 TOPOLOGY AND GEOMETRY.

Topological Motivation from Metric Spaces, surfaces etc. Topological spaces; Definition, Diverse Examples . Open sets, closed sets, closure, interior, boundary. Continuous maps. Existence of continuous functions. For metric spaces there are 'sufficiently many'. Convergence of sequences of functions.

Subspace Topology, Product Topology, and Identification Topology; Examples of identification Spaces.

Compactness and Connectedness. Compact sets and finite sets. Compactness and Continuity. Tychonoff's Theorem (without proof). Connectedness, path connectedness. Examples.

Homotopy of paths. The Fundamental group. Some Constructions. Homotopy type. Covering Spaces.

Triangulations, Barycentric Subdivision, Simplicial Complexes. Simplicial Approximation Theorem.

Suggested References:

- **James R. Munkres. Topology.**
- **M.A. Armstrong. Basic Topology.**
- **Singer and Thorpe. Lecture Notes on Elementary Topology and Geometry.**
- **https://people.maths.ox.ac.uk/ritter/masterclasses/ritter-lectures-on-geomery-and-topology.pdf**

VI.3 MACHINE LEARNING.

Basic concepts: supervised learning, unsupervised learning, reinforcement learning. Aspects of developing a learning system: training data, concept representation, function approximation. Discussion Sections: Linear Algebra, Probability, Vectorization

Decision tree learning: Representing concepts as decision trees. Recursive induction of decision trees. Picking the best splitting attribute: entropy and information gain. Searching for simple trees and computational complexity. Occam's razor. Overfitting, noisy data, and pruning.

Supervised learning: Linear Regression Logistic Regression, Perceptron, Exponential family Generative learning algorithms, Gaussian discriminant analysis MLE, MAP, Naive Bayes Support vector machines, and kernel methods k-Nearest Neighbors Practical machine learning advice: Bias/variance tradeoff and error analysis Learning Theory, Generalization errors + model selection, VC dimension Regularization and Model Selection Experimental evaluation of learning algorithms, cross-validation, learning curves, statistical hypothesis testing.

Deep Learning: NN architecture Forward/Back propagation

Unsupervised learning: Clustering. k-means, agglomerative clustering The EM Algorithm, Mixture of Gaussians. Principal Components Analysis, Dimensionality Reduction Independent components analysis

Ensemble learning: Boosting and Bagging

Reinforcement learning: Markov Decision Processes, Bellman equations Value iteration and policy iteration

References.

- Pattern Recognition and Machine Learning. C. Bishop.
- Elements of Statistical Learning. Hastie, Tibshirani, Friedman.
- CS229: Machine Learning by Dan Boneh and Andrew Ng.

VII.1 FUNCTIONAL ANALYSIS AND APPLICATIONS

Lebesgue Measure on $\mathbb{R}^d$: Outer Measure and its properties, Measurable sets, Lebesgue measure, σ-algebra of Borel sets. Measurable functions, Littlewood's three Principles. The Lebesgue Integral, Convergence Theorems, Fubini's Theorem.

L^p Spaces and Banach Spaces: Holder and Minkowski Inequalities. Completeness of L^p Spaces. Banach Spaces, Examples. Dual of a Banach Space, weak convergence. Hahn Banach Theorem. Bounded Linear Operators on a banach Space.

Baire Category Theorem. Uniform Boundedness Theorem, Open Mapping Theorem, Closed Graph Theorem.

Hilbert Spaces: Inner product, Orthogonality. Orthonormal Bases, Examples. Closed Subspaces, Projections, Riesz Representation Theorem. Spectral Theorem for Compact Normal Operators.

Suggested References.

- **E. Stein and R. Shakarchi. Real Analysis.**
- **E. Stein and R. Shakarchi. Functional Analysis.**

  (The course is based on selected Chapters from the above two books.)

- **G. Simmons. Introduction to Topology and Modern Analysis.**
- **J.B. Conway. A course in Functional Analysis.**